Electrostatics and simply connectedness

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People sometimes mention (correctly) that $\nabla \times \boldsymbol{E} = \mathbf{0}$ only implies $\boldsymbol{E} = -\nabla\phi$ in a region that is simply connected. A simply connected domain is one in which any loop can be contracted to a point without any part of the loop leaving the domain. For example, a solid ball is simply connected, while a solid torus (donut with a hole) is not.

The argument for the electrostatic potential in a simply connected domain goes like this: We have the Maxwell equation $\nabla \times \boldsymbol{E} = \mathbf{0}$ in our domain. Integrate $\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l}$ around any loop contained in your domain and apply Stokes's theorem to find $\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \int (\nabla \times \boldsymbol{E}) \cdot \mathrm{d}\boldsymbol{S} = 0$. That's the step we can *only* take if the domain is simply connected; if it's not, then for some loops, the surfaces they are the boundary of necessarily extend outside the domain, where you can't say $\nabla \times \boldsymbol{E} = \mathbf{0}$. Now, the actual 3D space (or 4D spacetime) we live in *is* simply connected, as far as we can tell. Thus we *do* have $\oint \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l} = 0$ for any closed loop. This implies that $\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$ from $\boldsymbol{r}_0$ to $\boldsymbol{r}$ is independent of the path taken. Thus we can explicitly construct a well-defined function $\phi(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}$, where $\boldsymbol{r}_0$ is some arbitrary fixed point in the domain. Considering a small change in $\boldsymbol{r}$, we find $\boldsymbol{E} = -\nabla\phi$.

Ok... but suppose you have an electrostatics problem where the domain is not simply connected? E.g. take a hollow donut with conducting walls, put some charges in the toroidal cavity, and try to calculate $\boldsymbol{E}$ in the cavity. Can we really not use an electric potential to solve this problem!?

Actually we can. What we do is use the knowledge that whatever domain we're actually considering is in fact just some subdomain of 3D space which has $\nabla \times \boldsymbol{E} = \mathbf{0}$ defined everywhere, and is simply connected, and so has $\boldsymbol{E} = -\nabla\phi$ everywhere. We thus go ahead and just use the electric potential even in a problem where the region of interest is multiply connected. Doing this actually excludes various $\boldsymbol{E}$ fields which might otherwise be allowed. For example, suppose we lived in a universe that was 3D space except with an infinite cylinder along the z-axis removed. Such a universe is then not simply connected. Even if Maxwell's equations held in that universe, you could have an azimuthal $\boldsymbol{E}$ field with $|\boldsymbol{E}| = \text{const}/r$, where r is radial distance from the z-axis. Such a field has zero divergence and zero curl everywhere in that universe, so nothing in Maxwell's equations forbids it. Charges could accelerate endlessly in azimuthal loops, and there would not exist an electrostatic potential in general. In other words; things would be very different even though Maxwell's equations hold! So actually we lean on the simply connectedness of our universe all the time, even when solving problems in multiply connected domains, which is interesting I think :)